

# Bipartite Induced Subgraphs and Well-Quasi-Ordering\*

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## Abstract

We study bipartite graphs partially ordered by the induced subgraph relation. Our goal is to distinguish classes of bipartite graphs which are or are not well-quasi-ordered (wqo) by this relation. Answering an open question from [3], we prove that  $P_7$ -free bipartite graphs are not wqo. On the other hand, we show that  $P_6$ -free bipartite graphs are wqo. We also obtain some partial results on subclasses of bipartite graphs defined by forbidding more than one induced subgraph.

*Keywords:* Bipartite graph; Induced subgraph; Well-quasi-ordering

## 1 Introduction

A binary relation  $\leq$  on a set  $X$  is a *quasi-order* if it is reflexive and transitive. Two elements  $x, y \in X$  are said to be incomparable if neither  $x \leq y$  nor  $y \leq x$ . An *antichain* in a quasi-order is a set of pairwise incomparable elements. A quasi-order  $(X, \leq)$  is a *well-quasi-order* if  $X$  contains no infinite strictly decreasing sequences and no infinite antichains.

In this paper, we study binary relations defined on sets of graphs. A graph  $H$  is said to be a *minor* of a graph  $G$  if  $H$  can be obtained from  $G$  by a (possibly empty) sequence of vertex deletions, edge deletions and edge contractions. According to the celebrated Graph Minor Theorem of Robertson and Seymour, the set of all graphs is well-quasi-ordered by the graph minor relation [8]. This, however, is not the case for the more restrictive relations such as subgraphs or induced subgraphs. A graph  $H$  is a *subgraph* of  $G$  if  $H$  can be obtained from  $G$  by a (possibly empty) sequence of vertex deletions and edge deletions;  $H$  is an *induced subgraph* of  $G$  if  $H$  can be obtained from  $G$  by a (possibly empty) sequence of vertex deletions. Clearly, the cycles  $C_3, C_4, C_5, \dots$  form an infinite antichain with respect to both relations. Except for this example, only a few other infinite antichains are known with respect to the subgraph or induced subgraph relations. One of them is the sequence of graphs  $H_1, H_2, H_3, \dots$  represented in Figure 1(left). Moreover, Ding proved in [3] that, in a sense, the cycles  $C_3, C_4, C_5, \dots$  and the graphs  $H_1, H_2, H_3, \dots$  are the only two infinite antichains with respect to the subgraph relation. More formally, Ding proved that a class of graphs closed under taking subgraphs is well-quasi-ordered by the subgraph relation if and only if it contains finitely many graphs  $C_n$  and  $H_n$ . The situation with induced subgraphs is less explored.

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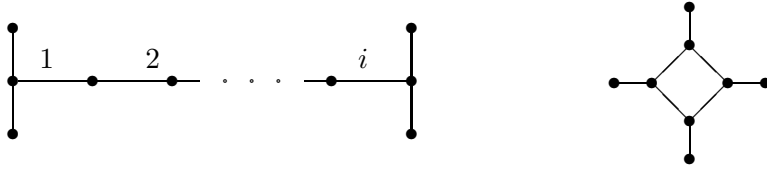


Figure 1: Graphs  $H_i$  (left) and  $Sun_4$  (right)

Damaschke [2] proved that the class of cographs is well-quasi-ordered by induced subgraphs. A cograph is a graph whose every induced subgraph with at least two vertices is either disconnected or the complement of a disconnected graph. The class of cographs is precisely the class of  $P_4$ -free graphs, i.e., graphs containing no  $P_4$  as an induced subgraph. In [3], Ding studied bi-cographs, i.e., the bipartite analog of cographs: these are bipartite graphs whose every induced subgraph with at least two vertices is either disconnected or the *bipartite complement* of a disconnected graph. Ding proved that the class of bi-cographs is also well-quasi-ordered by induced subgraphs. In terms of forbidden induced subgraphs this is precisely the class of  $(P_7, Sun_4, S_{1,2,3})$ -free bipartite graphs [3] (see also [5]), where  $Sun_4$  is the graph represented in Figure 1(right) and  $S_{1,2,3}$  is a tree with 3 leaves being of distance 1,2,3 from the only vertex of degree 3.

Obviously, exclusion of an induced path is a necessary condition for a class of graphs defined by finitely many forbidden induced subgraphs to be well-quasi-ordered, since otherwise the class contains infinitely many cycles. It is also necessary for such classes to exclude the complement of an induced path, since the complements of cycles also form an antichain with respect to the induced subgraph relation. In the case of bipartite graphs, together with an induced path one also has to exclude the *bipartite complement*  $\tilde{P}_k$  of an induced path  $P_k$ . Excluding an induced path and the bipartite complement of an induced path is not, however, sufficient for a class of bipartite graphs to be well-quasi-ordered. In [3], Ding found an infinite antichain of  $(P_8, \tilde{P}_8)$ -free bipartite graphs. On the other hand, he proved that  $(P_6, \tilde{P}_6)$ -free bipartite graphs are well-quasi-ordered by induced subgraphs. Observe that the bipartite complement of a  $P_7$  is a  $P_7$  again. The question whether the class of  $P_7$ -free bipartite graphs is well-quasi-ordered remained open for about 20 years. In the present paper we answer this question negatively by exhibiting an antichain of  $P_7$ -free bipartite graphs. Moreover, we show that this antichain is also  $Sun_4$ -free. On the other hand, we show that  $(P_7, Sun_1)$ -bipartite graphs are well-quasi-ordered by the induced subgraph relation, where  $Sun_1$  is the graph obtained from  $Sun_4$  by deleting 3 vertices of degree 1. We also obtain two other positive results. First, we show that  $(P_7, S_{1,2,3})$ -free bipartite graphs are well-quasi-ordered by induced subgraphs, generalizing both the bi-cographs and  $P_6$ -free graphs. Second, we prove that  $P_k$ -free bipartite permutation graphs are well-quasi-ordered by induced subgraphs for any value of  $k$ . The latter fact is in contrast with one more negative result of the present paper: by strengthening the Ding's idea, we show that  $(P_8, \tilde{P}_8)$ -free bipartite graphs are not well-quasi-ordered even when restricted to *biconvex graphs*, a class generalizing bipartite permutation graphs. The relationship between the classes of graphs under consideration is represented in Figure 2.

All graphs in this paper are undirected, without loops or multiple edges. The vertex set

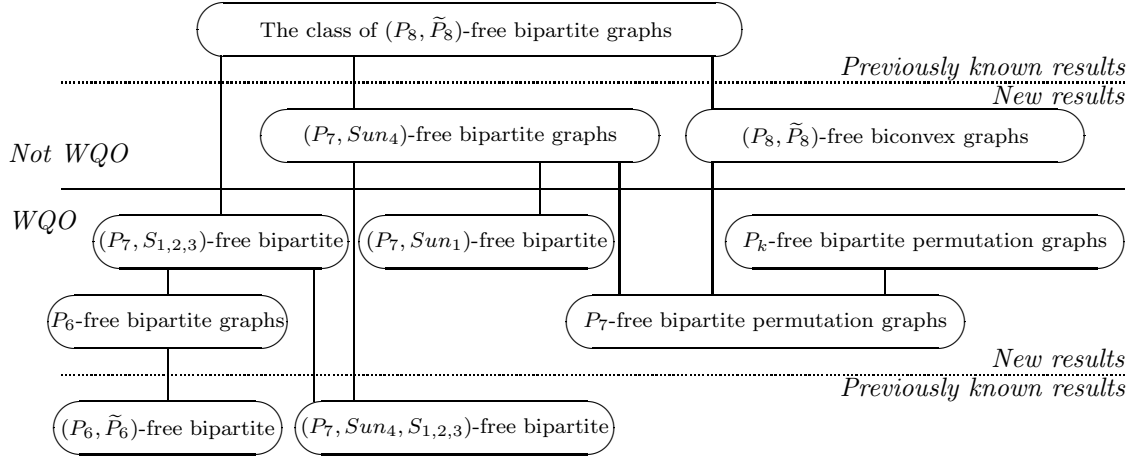


Figure 2: Inclusion relationships between subclasses of bipartite graphs

of a graph  $G$  is denoted  $V(G)$  and its edge set  $E(G)$ . For a subset  $U \subseteq V(G)$ , by  $G[U]$  we denote the subgraph of  $G$  induced by  $U$ . The neighborhood of a vertex  $v \in V(G)$  (i.e., the set of vertices of  $G$  adjacent to  $v$ ) is denoted  $N_G(v)$ . The degree of a vertex is the number of its neighbors. A graph is 1-regular if each of its vertices has degree 1.

As usual, we denote by  $P_n$ ,  $C_n$  and  $K_n$  the chordless path, the chordless cycle and the complete graph on  $n$  vertices. Also,  $2K_2$  is the disjoint union of two copies of  $K_2$ .

A graph is bipartite if the vertex set of the graph can be split into two parts each of which is an independent set, i.e., a set of pairwise nonadjacent vertices. The bipartite complement of a bipartite graph  $G = (V_1, V_2, E)$  with parts  $V_1$  and  $V_2$  and vertex set  $E$  is a bipartite graph  $\tilde{G} = (V_1, V_2, V_1 \times V_2 - E)$ .

We say that a graph  $G$  is  $H$ -free if  $G$  contains no copy of  $H$  as an induced subgraph. It is well known (and not difficult to see) that a  $2K_2$ -free bipartite graph possesses the property that the vertices in each part of the graph can be linearly ordered under inclusion of their neighborhoods.

## 2 Not well-quasi-ordered classes of bipartite graphs

In [3], Ding proved that the class of  $(P_8, \tilde{P}_8)$ -free bipartite graphs is not well-quasi-ordered by the induced subgraph relation. In this section, we strengthen this result in two ways. First, we show that  $P_7$ -free bipartite graphs are not wqo. Then we prove that  $(P_8, \tilde{P}_8)$ -free *biconvex* graphs are not wqo. To prove the results, in both cases we use the notion of a permutation, i.e., a bijection of the set  $[n] := \{1, 2, \dots, n\}$  to itself. To represent a permutation  $\pi : [n] \rightarrow [n]$ , we use one of the following two ways:

- one-line notation, which is the ordered sequence  $(\pi(1), \pi(2), \dots, \pi(n))$ .
- a diagram (see Figure 3 for an example).

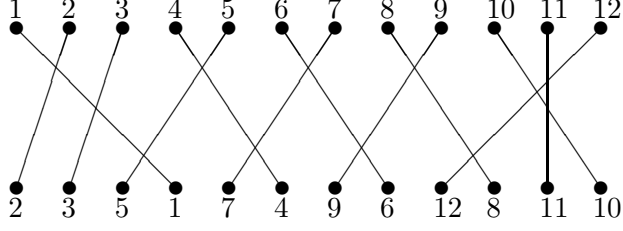


Figure 3: The diagram representing the permutation  $(2, 3, 5, 1, 7, 4, 9, 6, 12, 8, 11, 10)$ .

The permutation graph  $G_\pi$  of a permutation  $\pi$  is the intersection graph of the digram representing  $\pi$ . Figure 4 gives an example of a permutation and its permutation graph.

The composition  $\mu \circ \rho$  of two permutations  $\mu$  and  $\rho$  is a permutation  $\pi$  such that  $\pi(i) = \mu(\rho(i))$ . The inverse of a permutation  $\pi$  is a permutation  $\pi^{-1}$  such that  $\pi^{-1}(\pi(i)) = i$ .

Let  $\pi$  and  $\rho$  be two permutations given in one-line notation. We say that  $\pi$  is contained in  $\rho$  if  $\rho$  has a subsequence which is order-isomorphic to  $\pi$ . (Two sequences  $(a_1, \dots, a_n)$  and  $(b_1, \dots, b_n)$  are order-isomorphic if  $a_i \leq a_j$  if and only if  $b_i \leq b_j$ .) It is not difficult to see from the diagram representations that if  $G_\pi$  is not an induced subgraph of  $G_\rho$ , then  $\pi$  is not contained in  $\rho$ .

## 2.1 The class of $(P_7, \text{Sun}_4)$ -free bipartite graphs is not WQO

We start by introducing a special class of bipartite graphs defined as follows:

**Definition 1.** For each permutation  $\pi := \pi_n$  on  $[n]$ , the graph  $T := T_\pi$  is a bipartite graph with parts  $A \cup C$  and  $B \cup D$ , where:

1. The vertex set of  $T$  is the disjoint union of four independent vertex sets
  - $A := \{a_1, a_2, \dots, a_n\}$ ,
  - $B := \{b_1, b_2, \dots, b_n\}$ ,
  - $C := \{c_1, c_2, \dots, c_n\}$ ,
  - $D := \{d_1, d_2, \dots, d_n\}$ .
2.  $X(T) := T[A \cup B]$  is a 1-regular graph with  $e_i := a_i b_{\pi(i)}$  being an edge for each  $i \in [n]$ .
3.  $Y(T) := T[C \cup D]$  is a biclique (i.e., a complete bipartite graph).
4. Each of  $Z'(T) := T[A \cup D]$  and  $Z''(T) := T[B \cup C]$  is a  $2K_2$ -free bipartite graph defined as follows: for  $i = 1, 2, \dots, n$ ,
  - $N_{Z'}(a_i) = \{d_1, \dots, d_i\}$ ,
  - $N_{Z''}(b_i) = \{c_1, \dots, c_i\}$ .

Any graph of the form  $T_\pi$  will be called a  $T$ -graph.

In order to derive the main result of this section, we will show that every  $T$ -graph is  $(P_7, \text{Sun}_4)$ -free and that the set of  $T$ -graphs is not well-quasi-ordered by induced subgraphs. In fact, we will prove a slightly stronger result: we will show that every  $T$ -graph is  $(2P_3, \text{Sun}_4)$ -free, where  $2P_3$  is the graph obtained from  $P_7$  by deleting the central vertex.

**Lemma 1.** *Any  $T$ -graph is  $(2P_3, Sun_4)$ -free.*

*Proof.* Suppose, for contradiction, that  $T := T_\pi$  contains an induced  $2P_3$ . Then it is easy to check that each of the two  $P_3$  must contain at least one vertex in each of  $X(T)$  and  $Y(T)$ . Note that the vertices in  $2P_3 \cap Y(T)$  must all belong to the same part of the biclique  $Y(T)$ . We may assume without loss of generality that this part is  $D$ . It is clear that each  $P_3$  has an edge from  $A$  to  $D$ . But then  $Z'(T)$  is not  $2K_2$ -free, a contradiction showing that  $T$  is  $2P_3$ -free.

Now suppose, for contradiction, that  $T$  contains an induced  $Sun_4$ . Note that any two vertices in the same part of  $Y(T)$  have nested neighborhoods. Therefore, no two vertices of degree 3 in the  $Sun_4$  can belong to the same part of  $Y(T)$ . This implies that no two vertices of degree 3 in the  $Sun_4$  can belong to the same part of  $X(T)$ . Therefore, each of  $A, B, C$  and  $D$  must contain exactly one vertex of degree 3 in the  $Sun_4$ . Suppose that these vertices are  $a, b, c$  and  $d$ , respectively. The leaf attached to  $a$  in the  $Sun_4$  cannot belong to  $B$  (since otherwise  $a$  has degree more than 1 in  $X(T)$ ) and cannot belong to  $D$  (since otherwise  $Y(T)$  is not a biclique). This contradiction shows that  $T$  is  $Sun_4$ -free.  $\square$

Now we turn to showing that the set of  $T$ -graphs is not well-quasi-ordered by the induced subgraph relation. To this end, for each even  $n \geq 6$  we define a specific permutation  $\pi_n^*$ , as follows:

$$\pi_n^* := (4, 2, \dots, 2j, 2j - 5, \dots, n - 1, n - 3) \quad j = 3, \dots, n/2.$$

For instance,  $\pi_6^* = (4, 2, 6, 1, 5, 3)$  and  $\pi_8^* = (4, 2, 6, 1, 8, 3, 7, 5)$ . For  $n = 10$ , we use the diagram to represent  $\pi_{10}^*$  (see Figure 4 (left)). This diagram can also be interpreted as the subgraph  $X(T)$  of  $T_{\pi_{10}^*}$ , which can be seen by labeling the vertices in the upper part of the diagram by  $a_1, \dots, a_{10}$  consecutively from left to right and the vertices in the lower part of the diagram by  $b_1, \dots, b_{10}$  consecutively from left to right. The permutation graph  $G_{\pi_{10}^*}$  of the permutation  $\pi_{10}^*$  is represented in Figure 4 (right).

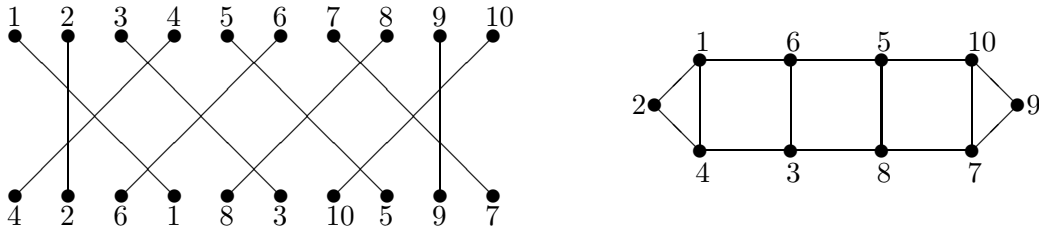


Figure 4: The permutation  $\pi_{10}^*$  (left) and the permutation graph  $G_{\pi_{10}^*}$  (right)

The important fact about the permutations  $\pi_n^*$  is that

**Claim 2.** *The sequence  $\pi_6^*, \pi_8^*, \pi_{10}^* \dots$  is an antichain of permutations with respect to the containment relation.*

This claim follows directly from the fact that no graph  $G_{\pi_n^*}$  is an induced subgraph of  $G_{\pi_m^*}$  with  $n \neq m$ , which can be easily seen. We now use Claim 2 in order to prove the following result.

**Lemma 3.** *The sequence  $T_{\pi_6^*}, T_{\pi_8^*}, T_{\pi_{10}^*}, \dots$  is an antichain with respect to the induced sub-graph relation.*

*Proof.* Suppose by contradiction that there is a graph  $H := T_{\pi_m^*}$  which is an induced subgraph of a graph  $G := T_{\pi_n^*}$  for some even  $6 \leq m < n$ . We fix an arbitrary embedding of  $H$  into  $G$ , i.e., we assume that  $V(H) \subset V(G)$ . We will denote the vertex subsets  $A, B, C, D$  of the graph  $H$  by  $A(H), B(H), C(H), D(H)$  and of the graph  $G$  by  $A(G), B(G), C(G), D(G)$ . Since both graphs are connected bipartite and  $\pi_n = \pi_n^{-1}$ , then in both graphs the role of the parts  $A \cup C$  and  $B \cup D$  is symmetric, so we may assume that

**Claim 4.**  $A(H) \cup C(H) \subseteq A(G) \cup C(G)$  and  $B(H) \cup D(H) \subseteq B(G) \cup D(G)$ .

Keeping Claim 4 in mind, we derive a series of conclusions. First, we show that

**Claim 5.**  $|A(H) \cap C(G)| \leq 1$ ,  $|B(H) \cap D(G)| \leq 1$ ,  $|C(H) \cap A(G)| \leq 1$ ,  $|D(H) \cap B(G)| \leq 1$ .

*Proof.* Suppose  $|A(H) \cap C(G)| \geq 2$ , and pick two distinct vertices  $a_i, a_j \in A(H)$  that belong to  $C(G)$ . Let  $\pi := \pi_m^*$ . Since  $Y(G)$  is a biclique, both  $b_{\pi(i)}$  and  $b_{\pi(j)}$  must lie in  $B(G)$ , which contradicts the  $2K_2$ -freeness of  $Z''(G)$ . Thus  $|A(H) \cap C(G)| \leq 1$ . The second inequality follows by symmetry.

Suppose  $c_i, c_j \in C(H) \cap A(G)$  ( $i < j$ ). Since  $b_j \in B(H)$  is adjacent to both  $c_i$  and  $c_j$ , we have  $b_j \in D(G)$ . Since  $|B(H) \cap D(G)| \leq 1$ ,  $b_i \in B(G)$ . Then  $a_{\pi^{-1}(i)}$  is adjacent to  $b_i$  but  $b_i$  has only one neighbor  $c_i$  in  $A(G)$ , and therefore  $a_{\pi^{-1}(i)} \in C(G)$ . Then  $a_{\pi^{-1}(i)} \in C(G)$  is adjacent to  $b_j \in D(G)$ , a contradiction. This proves that  $|C(H) \cap A(G)| \leq 1$ . The fourth inequality follows by symmetry. □

Now we prove that

**Claim 6.**  $|X(H) \cap Y(G)| \leq 1$  and  $|Y(H) \cap X(G)| \leq 1$ .

*Proof.* By Claim 5 and the definition of  $Y(G)$ , if the intersection  $X(H) \cap Y(G)$  contains two vertices, then these vertices must be adjacent. Let  $\pi := \pi_m^*$  and suppose an edge  $a_i b_{\pi(i)}$  of  $X(H)$  belongs to  $Y(G)$ . By Claim 5,  $|D(H) \cap B(G)| \leq 1$ , which means that  $a_i$  is adjacent to all but at most one vertex of  $D(H)$ . According to the definition of  $H$ , we conclude that  $i \in \{m-1, m\}$ . Similarly,  $b_{\pi(i)}$  is adjacent to all but at most one vertex of  $C(H)$ , implying that  $\pi(i) \in \{m-1, m\}$ . Together  $i \in \{m-1, m\}$  and  $\pi(i) \in \{m-1, m\}$  imply  $i = \pi(i) = m-1$ . From this and Claim 5 we conclude that both  $a_m \in A(H)$  and  $b_m \in B(H)$  belong to  $X(G)$ . Also, since

- $a_{m-1} \in A(H)$  belongs to  $Y(G)$ ,
- $a_{m-1}$  is not adjacent to  $d_m \in D(H)$  in  $H$  and
- $Y(G)$  is a biclique,

we conclude that  $d_m \in D(H)$  belongs to  $X(G)$ . Similarly,  $c_m \in C(H)$  belongs to  $X(G)$ . This contradicts the 1-regularity of  $X(G)$ , since  $c_m \in A(G)$  is adjacent to both  $b_m$  and  $d_m$  in  $B(G)$ . Thus  $|X(H) \cap Y(G)| \leq 1$ .

To prove the second inequality, suppose there is an edge  $cd$  of  $Y(H)$  belonging to  $X(G)$ . By definition, vertex  $c \in C(H)$  must have a neighbor in  $B(H)$ , and due to 1-regularity of

$X(G)$  this neighbor must belong to  $D(G)$ . Similarly,  $d$  must have a neighbor in  $A(H) \cap C(G)$ . But this contradicts  $|X(H) \cap Y(G)| \leq 1$ .  $\square$

Next, we show that

**Claim 7.**  $X(H) \cap Y(G) = Y(H) \cap X(G) = \emptyset$ .

*Proof.* Assume first that  $X(H) \cap Y(G)$  is not empty, and suppose without loss of generality that a vertex  $a_i$  of  $A(H)$  belongs to  $Y(G)$ . Then by Claim 6 all vertices of  $B(H)$  belong to  $X(G)$ . By Claim 5,  $|D(H) \cap B(G)| \leq 1$ , which means that  $a_i$  is adjacent to all but at most one vertex of  $D(H)$ . According to the definition of  $H$ , we conclude that  $i = m - 1$  or  $i = m$ . In either case, vertex  $b_m$  is not adjacent to  $a_i$ , and the neighborhood of  $b_m$  in the graph  $Z''(H)$  is strictly greater than the neighborhood of  $b_{\pi(i)}$ .

Suppose  $i = m$ . By Claim 5, at least one of  $c_{m-1}, c_m \in C(H)$  belongs to  $C(G)$ , say  $c_m \in C(G)$ . But then  $a_m, b_{m-3}, b_m, c_m$  induce a  $2K_2$ , contradicting the  $2K_2$ -freeness of  $Z''(G)$ .

Suppose now that  $i = m - 1$ . By definition, the vertex  $a_{m-1}$  of  $A(H)$  has a non-neighbor in  $D(H)$ . Therefore, the set  $D(H)$  must have a vertex in  $X(G)$ . This implies by Claim 6 that  $C(H) \subset C(G)$ , and hence the vertices  $a_{m-1}, b_{m-1}, c_m, b_m$  induce a  $2K_2$ , contradicting the  $2K_2$ -freeness of  $Z''(G)$ . This completes the proof of the fact that  $X(H) \cap Y(G) = \emptyset$ .

Now assume that  $Y(H) \cap X(G) \neq \emptyset$  and suppose without loss of generality that a vertex  $d_i$  of  $D(H)$  belongs to  $X(G)$ . Since  $X(H) \cap Y(G) = \emptyset$ , the vertex  $a_i \in A(H)$  lies in  $A(G)$  and has two neighbors  $b_{\pi(i)}$  and  $d_i$  in  $B(G)$ , a contradiction. Therefore,  $Y(H) \cap X(G) = \emptyset$ .  $\square$

Claims 7 and Claim 4 together imply the following conclusion.

**Claim 8.**  $A(H) \subseteq A(G)$ ,  $B(H) \subseteq B(G)$ ,  $C(H) \subseteq C(G)$  and  $D(H) \subseteq D(G)$ .

Assuming that  $H$  is an induced subgraph of  $G$ , we must conclude that the ordering of vertices of  $A(H)$  respects the ordering of vertices of  $A(G)$ , and similarly, the ordering of vertices of  $B(H)$  respects the ordering of vertices of  $B(G)$ . But then we must conclude that  $\pi_m^*$  is contained in  $\pi_n^*$  which is a contradiction to Claim 2. This contradiction completes the proof of the lemma.  $\square$

Lemmas 1 and 3 imply the main result of this section.

**Theorem 9.** *The class of  $(P_7, \text{Sun}_4)$ -free bipartite graphs is not well-quasi-ordered by the induced subgraph relation.*

## 2.2 The class of $(P_8, \tilde{P}_8)$ -free biconvex graphs is not WQO

A bipartite graph is *biconvex* if the vertices of the graph can be linearly ordered so that the neighborhood of each vertex forms an interval, i.e., the neighborhood consists of consecutive vertices in the order. [9] Strengthening the result from [3], we show in this section that the class of  $(P_8, \tilde{P}_8)$ -free *biconvex* graphs is not wqo by the induced subgraph relation. We start by introducing two special types of permutations.

**Definition 2.** A permutation  $\pi_n$  is *convex* if for any  $1 \leq i \leq n$  the set  $\pi_n^{-1}(\{i, i+1, \dots, n-1, n\})$  forms an interval, i.e., the elements of the set occupy consecutive positions in the permutation.

For instance, the permutation  $\rho = (1, 2, 3, 5, 7, 9, 10, 8, 6, 4)$  is convex. Indeed, the elements of the set  $\{5, 6, 7, 8, 9, 10\}$  occupy positions 4, 5, 6, 7, 8, 9, the elements of the set  $\{6, 7, 8, 9, 10\}$  occupy positions 5, 6, 7, 8, 9, and the same is true for any other set of the form  $\{i, i+1, \dots, n-1, n\}$ . The permutation  $\mu = (2, 3, 5, 7, 10, 9, 8, 6, 4, 1)$  is another example of a convex permutation.

**Definition 3.** A permutation  $\pi$  is *biconvex* if there are two convex permutations  $\mu$  and  $\rho$  such that  $\pi = \mu \circ \rho^{-1}$ .

To give an example, consider the following permutation:  $\pi = (2, 3, 5, 1, 7, 4, 10, 6, 9, 8)$ . It is not difficult to verify that  $\pi = \mu \circ \rho^{-1}$ , where  $\mu$  and  $\rho$  are the two convex permutations given above. For instance,  $\pi(1) = \mu(\rho^{-1}(1)) = 2$ ,  $\pi(2) = \mu(\rho^{-1}(2)) = 3$ ,  $\pi(3) = \mu(\rho^{-1}(3)) = 5$ , etc. Therefore,  $\pi$  is a biconvex permutation.

By  $\pi[\mu, \rho]$  we shall denote a biconvex permutation  $\pi$  given together with a pair of convex permutations  $\mu$  and  $\rho$  such that  $\pi = \mu \circ \rho^{-1}$ . Now we introduce a special class of bipartite graphs defined as follows:

**Definition 4.** For a biconvex permutation  $\pi := \pi_n[\mu, \rho]$ , the graph  $S := S_\pi$  is a bipartite graph with parts  $A \cup C$  and  $B$ , where:

1.  $V(S)$  is the disjoint union of three independent vertex sets
  - $A := \{a_1, a_2, \dots, a_n\}$ ,
  - $B := \{b_1, b_2, \dots, b_n\}$ ,
  - $C := \{c_1, c_2, \dots, c_n\}$ ,
2. Each of  $X(S) := S[A \cup B]$  and  $Y(S) := S[B \cup C]$  is a  $2K_2$ -free bipartite graph defined as follows: for  $i = 1, 2, \dots, n$ ,
  - $N_X(b_i) = \{a_1, \dots, a_{\rho(i)}\}$ ,
  - $N_Y(b_i) = \{c_1, \dots, c_{\mu(i)}\}$ .

Any graph of the form  $S_\pi$  will be called an  $S$ -graph.

*Remark.*  $N_G(a_i) = \{b_{\rho^{-1}(i)}, \dots, b_{\rho^{-1}(n)}\}$  and  $N_G(c_i) = \{b_{\mu^{-1}(i)}, \dots, b_{\mu^{-1}(n)}\}$ .

**Claim 10.** Any  $S$ -graph is a  $(P_8, \tilde{P}_8)$ -free biconvex graph.

*Proof.* Let  $S := S_\pi$  be an  $S$ -graph associated with a biconvex permutation  $\pi := \pi_n$  such that  $\pi = \mu \circ \rho^{-1}$ , where  $\mu$  and  $\rho$  are two convex permutations. The  $(P_8, \tilde{P}_8)$ -freeness of  $S$  follows from the  $2K_2$ -freeness of  $X(T)$  and  $Y(T)$ . Now let us prove that  $S$  is biconvex. To this end, we need to show that the vertices in each part of the graph can be linearly ordered so that the neighborhood of any vertex in the opposite part forms an interval. To achieve this goal we keep the natural order of the vertices in the  $B$ -part, i.e.,  $B = (b_1, \dots, b_n)$ . The vertices of the  $A \cup C$ -part are ordered under inclusion of their neighborhoods, increasingly for the  $A$ -vertices



and decreasingly for the  $C$ -vertices, i.e., the vertices with the largest neighborhood in  $A$  and  $C$  are in the middle of the order. Now let us show that the defined order is biconvex.

Let  $b$  be any vertex from  $B$ . If  $b$  is adjacent to any vertex  $a$  from  $A$ , then  $b$  is adjacent to any vertex from  $A$  with larger neighborhood than  $N(a)$ , i.e.,  $b$  is adjacent to any vertex of  $A$  following  $a$ . Similarly, if  $b$  is adjacent to any vertex  $c$  from  $C$ , then  $b$  is adjacent to any vertex from  $C$  with larger neighborhood than  $N(c)$ , i.e.,  $b$  is adjacent to any vertex of  $C$  preceding  $c$ . Therefore,  $N(b)$  is an interval.

Now let  $a_i$  be a vertex from  $A$ . Let  $I$  be the interval (i.e., the set of positions) of length  $n - i + 1$  containing the elements  $\{i, \dots, n\}$  of the permutation  $\rho$ . Then  $N(a_i) = \{b_j : j \in I\}$ , i.e.,  $N(a_i)$  is an interval. Similarly, if  $c_i$  is a vertex from  $C$  and  $I$  is the interval of length  $n - i + 1$  containing the elements  $\{i, \dots, n\}$  of the permutation  $\mu$ , then  $N(c_i) = \{b_j : j \in I\}$ , i.e.,  $N(c_i)$  is an interval  $\square$

Now we define a specific permutation  $\pi_n^*$  in the following way: for each even  $n \geq 8$ ,

$$\pi_n^* := (2, 3, 5, 1, \dots, 2j + 3, 2j, \dots, n, n - 4, n - 1, n - 2) \quad j = 2, \dots, n/2 - 4.$$

For instance,  $\pi_8^* = (2, 3, 5, 1, 8, 4, 7, 6)$  and  $\pi_{10}^* = (2, 3, 5, 1, 7, 4, 10, 6, 9, 8)$ . The permutation  $\pi_{12}^*$  is represented in Figure 3.

Let us show that  $\pi_n^*$  is a biconvex permutation. To this end, we define two convex permutations  $\rho_n^*$  and  $\mu_n^*$  in the following way:

$$\rho_n^* := (1, 2, 3, 5, \dots, \text{odd numbers}, \dots, n - 3, n - 1, n, n - 2, \dots, \text{even numbers}, \dots, 6, 4).$$

$$\mu_n^* := (2, 3, 5, \dots, \text{odd numbers}, \dots, n - 3, n, n - 1, n - 2, n - 4, \dots, \text{even numbers}, \dots, 6, 4, 1).$$

It is not difficult to verify that for  $n = 10$  the permutations  $\pi_n^*$ ,  $\rho_n^*$  and  $\mu_n^*$  coincide with the permutations  $\pi$ ,  $\rho$  and  $\mu$  defined in the beginning of the section.

**Claim 11.**  $\pi_n^* = \mu_n^* \circ \rho_n^{*-1}$ .

*Proof.* For small and large values of  $i$ , one can verify by direct inspection that  $\pi_n^*(i) = \mu_n^*(\rho_n^{*-1}(i))$ . Now let  $4 < i < n - 3$ . If  $i$  is odd then  $\pi_n^*(i) = \mu_n^*(\rho_n^{*-1}(i)) = i + 2$ , and if  $i$  is even then  $\pi_n^*(i) = \mu_n^*(\rho_n^{*-1}(i)) = i - 2$ .  $\square$

**Lemma 12.** *The sequence  $S_{\pi_8^*}, S_{\pi_{10}^*}, S_{\pi_{12}^*}, \dots$  is an antichain with respect to the induced subgraph relation.*

*Proof.* Suppose by contradiction that there is a graph  $H := S_{\pi_m^*}$  which is an induced subgraph of a graph  $G := S_{\pi_n^*}$  for some even  $8 \leq m < n$ . We fix an arbitrary embedding of  $H$  into  $G$ , i.e., we assume that  $V(H) \subset V(G)$ . Since both graphs are connected bipartite, we may assume that exactly one of the following two possibilities holds:

1.  $A(H) \cup C(H) \subseteq A(G) \cup C(G)$  and  $B(H) \subseteq B(G)$ .
2.  $A(H) \cup C(H) \subseteq B(G)$  and  $B(H) \subseteq A(G) \cup C(G)$

We claim that the first possibility holds.

**Claim 13.**  $A(H) \cup C(H) \subseteq A(G) \cup C(G)$  and  $B(H) \subseteq B(G)$ .

*Proof.* Note that, by definition,  $A(G) \cup C(G)$  can be partitioned into two chains with respect to the neighborhood inclusion. On the other hand, the set  $B(H)$  does not have this property, since  $b_{n/2}, b_{n/2+1}, b_{n/2+2}$  is an antichain of length 3 with respect to the same relation. Indeed,  $\rho^*(n/2) = n - 3$ ,  $\rho^*(n/2 + 1) = n - 1$ ,  $\rho^*(n/2 + 2) = n$  and  $\mu^*(n/2) = n$ ,  $\mu^*(n/2 + 1) = n - 1$  and  $\mu^*(n/2 + 2) = n - 2$ . This proves the claim.  $\square$

We make the following helpful remark:

- If two vertices of  $B(H)$  are incomparable with respect to the neighborhood inclusion in  $B(H)$ , then these two vertices must also be incomparable with respect to the neighborhood inclusion in  $B(G)$ .

Let  $B'(H)$  be the incomparability graph for the relation of neighborhood inclusion on the vertex set  $B(H)$ . In other words, two vertices of  $B(H)$  are adjacent in  $B'(H)$  precisely when they are incomparable with respect to the neighborhood inclusion. We define  $B'(G)$  similarly.

Clearly, by the above remark,  $B'(H)$  must be a subgraph of  $B'(G)$ . But for any even  $n \geq 8$ , the graph  $B'(S_{\pi_n^*})$  is simply the permutation graph  $G_{\pi_n^*}$  of  $\pi_n^*$  and this graph is represented in Figure 5.

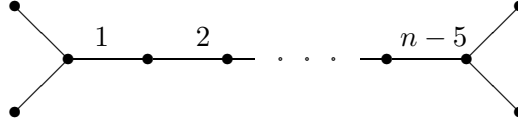


Figure 5: The graph  $B'(S_{\pi_n^*}) = G_{\pi_n^*}$

It is not difficult to see that the sequence of graphs  $G_{\pi_n^*}$ ,  $n \geq 8$ , forms an antichain with respect to the (induced) subgraph relation. Therefore,  $B'(H)$  is not a subgraph of  $B'(G)$ . As a result,  $H$  is not an induced subgraph of  $G$ . This contradiction completes the proof of Lemma 12.  $\square$

Lemma 12 and Claim 10 together imply the main result of this section:

**Theorem 14.** *The class of  $(P_8, \tilde{P}_8)$ -free biconvex graphs is not well-quasi-ordered by the induced subgraph relation.*

### 3 Well-quasi-ordered classes of bipartite graphs

In this section, we turn to positive results, i.e., to classes of bipartite graphs which are well-quasi-ordered by the induced subgraph relation.

### 3.1 The class of $(P_7, S_{1,2,3})$ -free bipartite graphs

In [3], Ding showed that  $(P_7, S_{1,2,3}, Sun_4)$ -free bipartite graphs and  $(P_6, \tilde{P}_6)$ -free bipartite graphs are well-quasi-ordered by the induced subgraph relation. Now we extend both results to the larger class of  $(P_7, S_{1,2,3})$ -free bipartite graphs. To this end, let us introduce the following notation.

Given a set of bipartite graphs  $\mathcal{F}$ , we denote by  $[\mathcal{F}]$  the set of bipartite graphs constructed from graphs in  $\mathcal{F}$  by means of the following three binary operations defined for any two disjoint bipartite graphs  $G_1 = (X_1, Y_1, E_1)$  and  $G_2 = (X_2, Y_2, E_2)$ :

- the disjoint union is the operation that creates out of  $G_1$  and  $G_2$  the bipartite graph  $G = (X_1 \cup X_2, Y_1 \cup Y_2, E_1 \cup E_2)$ ,
- the join is the operation that creates out of  $G_1$  and  $G_2$  the bipartite graph which is the bipartite complement of the disjoint union of  $\tilde{G}_1$  and  $\tilde{G}_2$ ,
- the skew join is the operation that creates out of  $G_1$  and  $G_2$  the bipartite graph  $G = (X_1 \cup X_2, Y_1 \cup Y_2, E_1 \cup E_2 \cup \{xy : x \in X_1, y \in Y_2\})$ .

The importance of these operations is due to the following theorem.

**Theorem 15.** *If  $\mathcal{F}$  is a set of bipartite graphs well-quasi-ordered by the induced subgraph relation, then so is  $[\mathcal{F}]$ .*

For the proof of this theorem, we refer the reader to Theorems 4.1 and 4.4 from [3], where the author used this result (without formulating it implicitly) in his proof that  $(P_7, S_{1,2,3}, Sun_4)$ -free bipartite graphs and  $(P_6, \tilde{P}_6)$ -free bipartite graphs are well-quasi-ordered by the induced subgraph relation. Now we combine Theorem 15 with a result from [4] that can be formulated as follows.

**Theorem 16.** *The class of  $(P_7, S_{1,2,3})$ -free bipartite graphs is precisely  $[\{K_1\}]$ .*

Together, Theorem 15 and Theorem 16 imply the following conclusion.

**Theorem 17.** *The class of  $(P_7, S_{1,2,3})$ -free bipartite graphs is well-quasi-ordered by the induced subgraph relation.*

### 3.2 The class of $(P_7, Sun_1)$ -free bipartite graphs

The graph  $Sun_1$  is obtained from  $Sun_4$  (Figure 1) by deleting three vertices of degree 1. Therefore, the class of  $(P_7, Sun_1)$ -free bipartite graphs is a proper subclass of  $(P_7, Sun_4)$ -free bipartite graphs. In contrast to the result of Section 2.1, below we prove that  $(P_7, Sun_1)$ -free bipartite graphs are well-quasi-ordered by the induced subgraph relation. According to Theorem 15, it suffices to show that the set of *connected*  $(P_7, Sun_1)$ -free bipartite graphs is well-quasi-ordered by this relation. The following lemma shows that the structure of connected graphs in this class containing a  $C_4$  is rather simple.

**Lemma 18.** *Every connected  $(P_7, Sun_1)$ -free bipartite graph containing a  $C_4$  is complete bipartite.*

*Proof.* Let  $H$  be a  $(P_7, Sun_1)$ -free bipartite graph containing a  $C_4$ . Denote by  $H'$  any maximal complete bipartite subgraph of  $H$  containing the  $C_4$ . If  $H' \neq H$ , there must exist a vertex  $v$  outside  $H'$  that has a neighbor in  $H'$ . If  $v$  is adjacent to every vertex of  $H'$  in the opposite part, then  $H'$  is not maximal, and if  $v$  has a non-neighbor in the opposite part of  $H'$ , the reader can easily find an induced  $Sun_1$ . The contradiction in both cases shows that  $H' = H$ , i.e.,  $H$  is a complete bipartite graph.  $\square$

It is not difficult to see that there is no infinite antichain of complete bipartite graphs, which follows, for instance, from the fact that every complete bipartite graph is  $P_4$ -free and the class of  $P_4$ -free (not necessarily bipartite) graphs is well-quasi-ordered. This observation together with Lemma 18 reduces the problem from  $(P_7, Sun_1)$ -free bipartite graphs to  $(P_7, C_4)$ -free bipartite graphs. The proof that the class of  $(P_7, C_4)$ -free bipartite graphs is well-quasi-ordered is based on the following lemma.

**Lemma 19.** *No  $(P_7, C_4)$ -free bipartite graph contains  $P_9$  as a subgraph (not necessarily induced).*

*Proof.* Let  $G$  be a  $(P_7, C_4)$ -free bipartite graph. To prove the lemma, we first derive the following helpful observation.

**Claim 20.** *If  $P := (a_1, a_2, \dots, a_7)$  is a copy of  $P_7$  contained in  $G$  as a subgraph, then  $P$  has exactly one chord in  $G$ , either  $a_1a_6$  or  $a_2a_7$ .*

*Proof.* Since  $G$  is  $P_7$ -free,  $P$  must contain a chord, and since  $G$  is bipartite, any chord of  $P$  connects an even-indexed vertex to an odd-indexed one. Among 6 possible chords of  $P$  only  $a_1a_6$  and  $a_2a_7$  do not produce a  $C_4$ , and these two chords cannot be present in the graph simultaneously, since otherwise the vertices  $a_1, a_2, a_7, a_6$  induce a  $C_4$ . Therefore,  $P$  must contain exactly one of  $a_1a_6$  or  $a_2a_7$  as a chord.  $\square$

Suppose now that  $Q := (b_1, b_2, \dots, b_9)$  is a copy of  $P_9$  contained as a subgraph in  $G$ , and for  $1 \leq i \leq 3$ , let  $Q_i := (b_i, b_{i+1}, \dots, b_{i+6})$ . If  $b_1b_6$  is a chord of  $Q$ , then Claim 20 applied to each of  $Q_1, Q_2$  and  $Q_3$  implies that  $Q$  contains exactly two chords, namely  $b_1b_6$  and  $b_3b_8$ . But then the vertices  $b_1, b_6, b_5, b_4, b_3, b_8, b_9$  induce a  $P_7$ , a contradiction.

The case when  $b_1b_6$  is not a chord of  $Q$  is symmetric and also leads (with the help of Claim 20) to an induced  $P_7$  in  $G$ . The contradiction in both cases shows that  $G$  does not contain  $P_9$  as a subgraph.  $\square$

Now we combine Lemma 19 with the following result by Ding [3].

**Theorem 21** (Ding [3]). *For any fixed  $k \geq 1$ , the class of graphs containing no  $P_k$  as a (not necessarily induced) subgraph is well-quasi-ordered by the induced subgraph relation.*

Together Lemma 19 and Theorem 21 imply the main conclusion of this section.

**Theorem 22.** *The class of  $(P_7, C_4)$ -free bipartite graphs is well-quasi-ordered by the induced subgraph relation.*

### 3.3 The class of $P_k$ -free bipartite permutation graphs

The class of bipartite permutation graphs is the intersection of bipartite graphs and permutation graphs. This class is a subclass of biconvex graphs (see e.g. [1]). In contrast to the result of Section 2.2 we show that  $P_k$ -free bipartite permutation graphs are well-quasi-ordered by the induced subgraph relation for any fixed value of  $k$ . In general, bipartite permutation graphs are not well-quasi-ordered by this relation, since they contain the antichain of graphs of the form  $H_i$  (Figure 1). Our proof is based on a number of known results.

Denote by  $H_{n,m}$  the graph with  $nm$  vertices which can be partitioned into  $n$  independent sets  $V_1 = \{v_{1,1}, \dots, v_{1,m}\}, \dots, V_n = \{v_{n,1}, \dots, v_{n,m}\}$  so that for each  $i = 1, \dots, n-1$  and for each  $j = 1, \dots, m$ , vertex  $v_{i,j}$  is adjacent to vertices  $v_{i+1,1}, v_{i+1,2}, \dots, v_{i+1,j}$  and there are no other edges in the graph. In other words, every two consecutive independent sets induce in  $H_{n,m}$  a universal chain graph. An example of the graph  $H_{n,n}$  with  $n = 5$  is given in Figure 6.

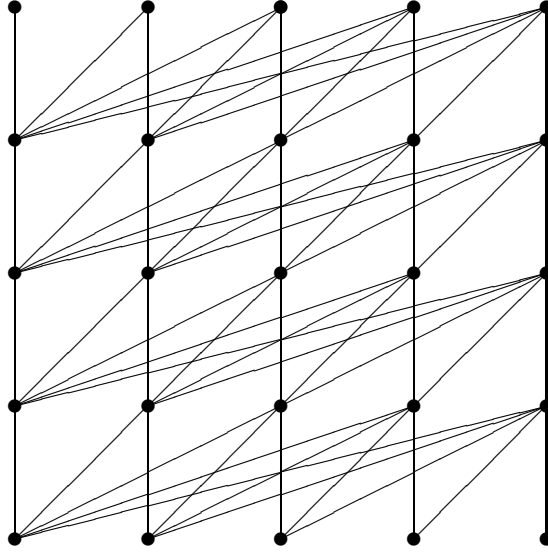


Figure 6: The graph  $H_{5,5}$

It is not difficult to see that the graph  $H_{n,n}$  is a bipartite permutation graph. Moreover, it was proved in [6] that  $H_{n,n}$  is an  $n$ -universal bipartite permutation graph in the sense that every bipartite permutation graph with  $n$  vertices is an induced subgraph of  $H_{n,n}$ . If a connected bipartite permutation graph is  $P_k$ -free, it occupies at most  $k$  consecutive levels of  $H_{n,n}$ . In other words, every connected  $P_k$ -free bipartite permutation graph is an induced subgraph of  $H_{k,n}$ .

In order to prove that  $P_k$ -free bipartite permutation graphs are well-quasi-ordered, we will show that any connected graph in this class is a  $k$ -letter graph. This notion was introduced in [7] and its importance for our study is due to the following result also proved in [7].

**Theorem 23.** *For any fixed  $k$ , the class of  $k$ -letter graphs is well-quasi-ordered by the induced subgraph relation.*

The  $k$ -letter graphs have been characterized in [7] as follows.

**Theorem 24** (Petkovšek [7]). *A graph  $G = (V, E)$  is a  $k$ -letter graph if and only if*

1. *there is a partition  $V_1, \dots, V_p$  of  $V(G)$  with  $p \leq k$  such that each  $V_i$  is either a clique or an independent set in  $G$ ,*
2. *there is a linear ordering  $L$  of  $V(G)$  such that for each pair of indices  $1 \leq i, j \leq p$ ,  $i \neq j$ , the intersection of  $E$  with  $V_i \times V_j$  is one of*
  - (a)  $L \cap (V_i \times V_j)$ ,
  - (b)  $L^{-1} \cap (V_i \times V_j)$ ,
  - (c)  $V_i \times V_j$ ,
  - (d)  $\emptyset$ .

**Corollary 25.** *Connected  $P_k$ -free bipartite permutation graphs are  $k$ -letter graphs.*

*Proof.* From Theorem 24 it follows that an induced subgraph of a  $k$ -letter graph is again a  $k$ -letter graph. In addition, we have seen already that any connected  $P_k$ -free bipartite permutation graph is an induced subgraph of  $H_{k,n}$ . Therefore, all we have to do is to prove that  $H_{k,n}$  is a  $k$ -letter graph. To this end, we define a partition  $V_1, \dots, V_k$  of the vertices of  $H_{k,n}$  by defining  $V_i$  to be the  $i$ -th row of  $H_{k,n}$ . Thus the first condition of Theorem 24 is satisfied. Then we define a linear ordering  $L$  of the vertices of  $H_{k,n}$  by listing first the vertices of the first column consecutively from bottom to top, then the vertices of the second column, and so on. Now let's take any two subsets  $V_i$  and  $V_j$  with  $i \neq j$ . If they are not consecutive rows of the graph, then the intersection of  $E$  with  $V_i \times V_j$  is empty. If they are consecutive, then the intersection of  $E$  with  $V_i \times V_j$  is either  $L \cap (V_i \times V_j)$  (if  $i > j$ ) or  $L^{-1} \cap (V_i \times V_j)$  (if  $i < j$ ). Thus the second condition of Theorem 24 is satisfied, which proves the corollary.  $\square$

Combining Corollary 25 with Theorems 15 and 23 we conclude that

**Corollary 26.** *For any fixed  $k$ , the class of  $P_k$ -free bipartite permutation graphs is well-quasi-ordered by the induced subgraph relation.*

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